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A stabilizer free weak Galerkin finite element method for elliptic equation with lower regularity

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Abstract. This paper presents error analysis of stabilizer free weak Galerkin finite element method (SFWG-FEM) for a second order elliptic equation with low regularity solutions. The standard error analysis of SFWG-FEM requires additional regularity on solutions, such as H^2-regularity for the second-order convergence. However, if the solutions are in H^{1+s} with $0 < s < 1$, numerical experiments show that the SFWG-FEM is also effective and stable with the $(1+s)$-order convergence rate, so we develop a theoretical analysis for it. We introduce a standard H^2 finite element approximation for the elliptic problem, and then we apply the SFWG-FEM to approach this smooth approximating finite element solution. Finally, we establish the error analysis for SFWG-FEM with low regularity in both discrete H^1-norm and standard L^2-norm. The $(P_k(T), P_{k-1}(e), [P_{k+1}(T)]^d)$ elements with dimensions of space $d = 2, 3$ are employed and the numerical examples are tested to confirm the theory.

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1. Introduction

The weak Galerkin finite element method (WG-FEM) is a useful numerical method for solving partial differential equations effectively. The WG-FEM is naturally derived from the standard finite element method (FEM) and the most important idea is to use the generalized functions and their weak derivatives which are defined as generalized distributions. The WG method is first introduced in [13, 14] by Wang and Ye for the second order elliptic equations, and a stabilizer term is added to WG-FEM in order to enforce the connection of discontinuous functions across element boundaries [7, 8]. Then the WG method finds applications in diverse areas including parabolic equations [24, 25, 27, 28], Stokes equations [10, 12], Maxwell equations [9, 11], biharmonic equation [26], Cahn-Hilliard-Cook

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equation [5], stochastic parabolic equations [29,30], eigenvalue problems [22,23], and so on. Nevertheless, the stabilizer makes the finite element formulations and programming complex. To remove the stabilizer term, a stabilizer free weak Galerkin finite element method (SFWG-FEM) is introduced in [16] by Ye and Zhang for the second order elliptic equations on polytopal meshes. The main idea of SFWG-FEM is raising the degree of polynomials for weak gradient computation to increase the connectivity of weak functions. The SFWG-FEM is firstly based on a weak gradient definition similar to the standard WG method, and for the element $(P_k(T), P_k(e), [P_j(T)]^d)$, it has been demonstrated to be reliable if $j \geq k+n-1$, where n represents the number of element's edges/faces [17]. Subsequently, the requirement $j \geq k+n-1$ has been loosened in [3]. Various configuration of $(P_k(T), P_l(e), [P_j(T)]^d)$ with $l, j \geq k$ leads to different schemes, cf. [1,20,21]. A novel definition of weak gradient is presented in [18] to the $(P_k(T), P_{k-1}(e), [P_{k+1}(T)]^d)$ element that employs fewer unknowns than in [17] while maintaining the same optimal order of convergence. Thereafter, SFWG-FEM is also successfully applied to several equations and reduces programming complexity [1–3, 17–21].

In order to reach the optimal convergence order for approximating the second order elliptic equations, in many published literature on WG-FEM and SFWG-FEM, the solution is usually assumed to have at least H^2-smoothness [1, 18]. As a result, it is demonstrated that the convergence rate are at least $\mathcal{O}(h)$ in H^1-norm and $\mathcal{O}(h^2)$ in L^2-norm with step-size h, respectively. However, numerical experiments show that we have the $\mathcal{O}(h^s)$ convergence in $|||\cdot|||$-norm and $\mathcal{O}(h^{1+s})$ convergence in L^2-norm when the exact solution has only H^{1+s}-regularity $(0 < s < 1)$. The $(1+s)$-order L^2 convergence analysis on WG-FEM is accomplished in [15], and the core concern in [15] is only on analysis but without numerical experiments. There is no such theoretical analysis for SFWG-FEM, so in this paper, we are devoted to studying the convergence analysis on SFWG-FEM with low regularity, and to proving the discrete H^1-norm and L^2-norm convergence rate to be $\mathcal{O}(h^s)$ and $\mathcal{O}(h^{1+s})$, respectively. The numerical examples are also tested to confirm the theory. Our strategy is divided into two steps. Firstly, we use H^2-regular Argyris elements [4] as the bases of FEM to approximate the second order elliptic equation whose solution has H^{1+s}-regularity, and then we get the optimal discrete H^1 and L^2 convergence order to be of $\mathcal{O}(h^s)$ and $\mathcal{O}(h^{1+s})$, respectively. Secondly, we utilize the SFWG-FEM to approximate the H^2-regular finite element solution, and then we have the $\mathcal{O}(h)$ convergence in $|||\cdot|||$-norm and $\mathcal{O}(h^2)$ convergence in L^2-norm, respectively. As a consequence, we arrive at the optimal convergence rates to be $\mathcal{O}(h^s)$ in $|||\cdot|||$-norm and $\mathcal{O}(h^{1+s})$ in L^2-norm, respectively, if the exact solution of the second order elliptic equation is only H^{1+s}-regular, which is an important supplementary for the new stabilizer free weak Galerkin finite element method theory.

The paper is organized as follows. In section 2, we apply the Argyris finite element approximation to a second order elliptic equation with H^{1+s} regularity, and provide an optimal order error estimate. In section 3, we introduce an SFWG-FEM and construct the corresponding scheme. In section 4, we perform an error analysis and prove the optimal convergence order for the SFWG-FEM. In section 5, we present some numerical experiments to validate the theoretical analysis. In section 6, we summarize the main

conclusions.

2. The variational form and H^2 finite element approximation

In this section, we introduce a second order elliptic equation with lower regularity, and then study its variational equation and an H^2 finite element approximation.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a convex bounded polygonal domain with Lipschitz continuous boundary $\partial\Omega$. Denote by $L^2(\Omega)$ and $L^2(\partial\Omega)$ the square integrable function spaces with inner products $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$, respectively. For $m \geq 1$, denote by $H^m = H^m(\Omega)$ the standard Sobolev space with norm $\|\cdot\|_m$ and by $\|\cdot\|$ the L^2-norm if no confusion occurs. Define by H_0^m a subspace of H^m with vanishing boundary values on $\partial\Omega$.

Consider a second order elliptic equation

$$\begin{aligned} -\nabla \cdot \big(A(x)\nabla u(x)\big) &= f(x), \quad in\ \Omega, \\ u &= 0, \quad on\ \partial\Omega. \end{aligned} \tag{2.1}$$

We assume

(H1) $f \in L^2(\Omega)$.

(H2) $A = A(x) = \big(a_{ij}(x)\big)_{d\times d} \in [W^{1,\infty}(\Omega)]^{d\times d}$ is a symmetric matrix-valued function in Ω and satisfies

$$\alpha_1\|\xi\|^2 \leq \xi^T A(x)\xi \leq \alpha_2\|\xi\|^2, \quad \forall \xi \in \mathbb{R}^d,\ x \in \Omega, \tag{2.2}$$

where ξ^T is the transpose of a column vector ξ and $\alpha_2 > \alpha_1 > 0$ are constants.

The variational formulation for Eq.(2.1) seeks $u \in H_0^1$ such that

$$(A\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1. \tag{2.3}$$

Under assumptions (H1) and (H2), the variational equation (2.3) possesses an unique solution $u \in H_0^1$, which is called a weak solution of (2.1) [4]. Throughout this paper, we will always use u to denote the weak solution of (2.1) when no confusion occurs.

For the second order elliptic equations, the solutions may not lie in H^2 if the boundary $\partial\Omega$ is piecewise smooth, or the function f and A have lower regularity. It is necessary to establish the error estimate for the problems with low regularity solutions, so we also assume

(H3) The weak solution u of (2.1) has H^{1+s} $(0 < s \leq 1)$ regularity with respect to $x \in \Omega$.

We denote by $\mathscr{T}_h$ a shape-regular conforming triangulation of the domain Ω (triangles if $d = 2$, tetrahedra if $d = 3$) satisfying the regularity requirements stated in [14]. Let $\mathscr{E}_h$ be the set of all edges or faces in $\mathscr{T}_h$, and $\mathscr{E}_h^0 = \mathscr{E}_h \backslash \partial\Omega$ be the set of all interior edges or faces. Let h_T be the diameter of each element $T \in \mathscr{T}_h$, and let $h = \max\limits_{T\in\mathscr{T}_h} h_T$. We take $d = 2$ as an example.

Let $S_h^0 \subset H_0^1 \cap H^2$ be a finite element space which is spanned by the Argyris bases [4], thus, all elements in S_h^0 are piecewise defined polynomials of degree no more than 5. The finite element approximation to (2.1) is to seek $\tilde{u}_h \in S_h^0$ such that

$$(A\nabla\tilde{u}_h, \nabla\tilde{v}_h) = (f, \tilde{v}_h), \quad \forall \tilde{v}_h \in S_h^0, \tag{2.4}$$

which is equivalent to seeking $\tilde{u}_h \in S_h^0$ such that

$$-P_h\big(\nabla \cdot (A\nabla\tilde{u}_h)\big) = P_h f, \tag{2.5}$$

where P_h is the orthogonal projection from $L^2(\Omega)$ to S_h^0. The classical finite element analysis for the second order elliptic equation gives that [6]

Lemma 2.1. *Assume (H1)-(H3) hold. Let $\tilde{u}_h \in S_h^0$ and $u \in H^{1+s}$ be the solution of* (2.1) *and* (2.4)*, respectively. Then the elliptical regularity inequality $\|\tilde{u}_h\|_2 \le C\|f\|$ holds, and for $0 < s \le 1$, there exists a constant $C > 0$ independent of h such that*

$$\begin{aligned} \|\nabla u - \nabla\tilde{u}_h\| &\le Ch^s\|f\|, \\ \|u - \tilde{u}_h\| &\le Ch^{1+s}\|f\|. \end{aligned}$$

3. Stabilizer free weak Galerkin finite element approximation

The primary idea behind WG-FEM is to approximate functions and their corresponding derivatives with the generalizations of functions and associated weak derivatives. For the WG methods with stabilizer term, the weak derivatives are defined in the sense of generalized distributions [13, 14]. To remove the stabilizer term, it is essential to increase the connectivity of a weak function across element boundary, thus, weak derivatives are approximated by polynomials of higher order [16, 17, 20].

In this section, we first introduce weak gradient operators and then perform stabilizer free weak Galerkin finite element approximation to (2.1). The corresponding L^2 projections which are essential to later analysis are also discussed.

3.1. Discrete weak gradient operator

A weak function on $T \in \mathscr{T}_h$ is denoted by a pair $v = \{v_0, v_b\}$ with $v_0 \in L^2(T)$ and $v_b \in L^2(\partial T)$. For any given integer $k \ge 1$, denote by $P_k(T)$ and $P_k(e)$ the sets of polynomials on $T \in \mathscr{T}_h$ and $e \in \mathscr{E}_h \cap \partial T$ with degree no more than k, respectively. Various configurations of $(P_l(T), P_m(e), [P_n(T)]^d)$ lead to different weak Galerkin finite element methods, and the $(P_r(T), P_{r-1}(e), [P_{r+1}(T)]^d)$ elements are introduced in our stabilizer free weak Galerkin finite element method.

For the integer $r \ge 1$, define a discrete weak function space $W_r(T)$ by

$$W_r(T) := \big\{w = \{w_0, w_b\} : w_0 \in P_r(T), w_b \in P_{r-1}(\partial T)\big\},$$

where $P_{r-1}(\partial T)$ is a piecewise polynomial set on ∂T that $P_{r-1}(\partial T) := \{w : w|_e \in P_{r-1}(e), \forall e \in \partial T \cap \mathscr{E}_h\}$. By patching over each element $T \in \mathscr{T}_h$, we construct the WG finite element space V_h as follows

$$V_h := \{v = \{v_0, v_b\} : v|_T \in W_r(T),\ \forall T \in \mathscr{T}_h\},$$

and let V_h^0 refer to the subspace of V_h that

$$V_h^0 := \{v = \{v_0, v_b\} : v \in V_h, v_b|_{\partial\Omega} = 0\}.$$

Define a space of vector-valued polynomials with degree no more than $r+1$ by

$$\mathscr{M}_h := \{q : q|_T \in [P_{r+1}(T)]^d,\ \forall T \in \mathscr{T}_h\}.$$

Next we introduce the definitions of new weak gradient operators.

Definition 3.1 (cf. Ye. X & Zhang S. [18]). *For $v \in W_r(T)$ on $T \in \mathscr{T}_h$, a local discrete weak gradient $\nabla_{w,r+1,T}$ is defined as a linear operator from $W_r(T)$ to $[P_{r+1}(T)]^d$ satisfying*

$$\left(\nabla_{w,r+1,T} v, q\right)_T = (\nabla v_0, q)_T + \langle Q_b(v_b - v_0), q \cdot \boldsymbol{n}\rangle_{\partial T},\ \forall\, q \in [P_{r+1}(T)]^d,$$

where $\boldsymbol{n}$ is a unit outward normal vector at ∂T and Q_b is the orthogonal projection from $L^2(\partial T)$ to $P_{r-1}(\partial T)$.

Definition 3.2 (cf. Ye. X & Zhang S. [18]). *A global discrete weak gradient $\nabla_{w,r+1}$ is defined as a linear operator from $V_h \cup H^1$ to $\mathscr{M}_h$ satisfying*

$$(\nabla_{w,r+1} v)|_T = \nabla_{w,r+1,T}(v|_T), \quad \forall v \in V_h \cup H^1.$$

To simplify notations, we will use ∇_w to denote both weak gradient operators defined above when no confusion occurs. For any function $v \in H^1(\Omega)$, we note that it could take $v_0 = v$ and $v_b = v$ in the preceding formulation. Then, for smooth functions, we notice the weak gradient becomes directly consistent with the strong gradient, that is, for any function $v \in H^1(\Omega)$,

$$(\nabla_w v, q) = (\nabla v, q), \quad \forall q \in [P_{r+1}(T)]^d.$$

3.2. SFWG approximation and corresponding L^2 projections

We define a bilinear operator $a(\cdot,\cdot)$ and the corresponding H^1-norm $|||\cdot|||$ for the SFWG algorithm of the second order elliptic equation (2.1) with the assumptions (H1)-(H2).

Definition 3.3 (cf. Ye. X & Zhang S. [18]). *For any $w, v \in V_h^0$, the bilinear operator $a(\cdot,\cdot)$ is defined as*

$$a(w,v) := \sum_{T \in \mathscr{T}_h} (A \nabla_w w, \nabla_w v)_T,$$

and equipped with the norm $|||\cdot|||$ in V_h^0 which is defined by $|||v|||^2 := a(v,v)$.

Thus, we obtain that the SFWG finite element approximation to (2.1) is seeking $u_h = \{u_0, u_b\} \in V_h^0$ such that

$$a(u_h, v_h) = (f, v_0), \quad \forall v_h = \{v_0, v_b\} \in V_h^0. \tag{3.1}$$

It follows from (2.2) that the bilinear operator $a(\cdot,\cdot)$ is bounded, and we have the following lemma.

Lemma 3.1. *Assume (H2), for any $w, v \in V_h^0$, there exists a constant $\beta > 0$ such that*

$$|a(w,v)| \leq \beta |||u||| |||v|||.$$

Next we introduce some useful L^2 projections and lemmas for later analysis.

Let Q_b be the orthogonal projection from $L^2(\partial T)$ to $P_{r-1}(\partial T)$ as we defined before, and Q_0 be the orthogonal projection from $L^2(T)$ to $P_r(T)$. Then, define the orthogonal projection $Q_h : H^1 \to V_h$ such that

$$(Q_h\phi)|_T := \{Q_0(\phi|_T), Q_b(\phi|_{\partial T})\}, \quad \forall \phi \in H^1.$$

In our case, denote by $\mathbb{Q}_h$ the L^2- orthogonal projection from $[L^2(T)]^d$ to $[P_{r+1}(T)]^d$, and denote by $\mathbb{Q}_k$ the orthogonal projection from $[L^2(T)]^d$ to $[P_k(T)]^d$ for integer $k \geq 0$. For these orthogonal projections, there hold the following properties.

Lemma 3.2 (cf. Wang J. & Ye. X Zhang S. [14]). *Let $\mathcal{T}_h$ be a finite element partition of Ω satisfying the shape regularity assumptions. Then, for any $\phi \in H^{k+1}$, we have*

$$\sum_{T\in\mathcal{T}_h} \|\phi - Q_0\phi\|_T^2 \leq C_1 h^{2(k+1)} \|\phi\|_{k+1}^2, \quad 0 \leq k \leq r,$$
$$\sum_{T\in\mathcal{T}_h} \left(\|\nabla\phi - \mathbb{Q}_l(\nabla\phi)\|_T^2\right) \leq C_2 h^{2k} \|\phi\|_{k+1}^2, \quad 0 \leq k \leq l,$$

where $C_1, C_2 > 0$ are constants independent of h.

Lemma 3.3 (cf. Ye. X & Zhang S. [18]). *For any $\phi \in H^1(T)$ on the element $T \in \mathcal{T}_h$, there holds the following commutative property*

$$\nabla_w \phi = \mathbb{Q}_h(\nabla\phi).$$

Remark 3.1. We would like to emphasize that the operator $\mathbb{Q}_h(\nabla\phi)$ is not identical with $\nabla_w Q_h$, and we can find that for any $\phi \in H^{r+1}$ the difference is

$$\begin{aligned}
(\nabla_w Q_h\phi, q)_T &= \big(\nabla(Q_0\phi), q\big)_T + \langle Q_b(Q_b\phi - Q_0\phi), q\cdot\boldsymbol{n}\rangle_{\partial T} \\
&= -(Q_0\phi, \nabla\cdot q)_T + \langle Q_0\phi, q\cdot\boldsymbol{n}\rangle_{\partial T} + \langle Q_b\phi - Q_b(Q_0\phi), q\cdot\boldsymbol{n}\rangle_{\partial T} \\
&= (\nabla\phi, q)_T - \langle\phi, q\cdot\boldsymbol{n}\rangle_{\partial T} + \langle Q_0\phi - Q_b(Q_0\phi), q\cdot\boldsymbol{n}\rangle_{\partial T} + \langle Q_b\phi, q\cdot\boldsymbol{n}\rangle_{\partial T} \\
&= (\mathbb{Q}_h(\nabla\phi), q) - \langle\phi - Q_b\phi, q\cdot\boldsymbol{n}\rangle_{\partial T} \\
&\quad + \langle Q_0\phi - Q_b(Q_0\phi), q\cdot\boldsymbol{n}\rangle_{\partial T}, \quad \forall q \in [P_{r+1}(T)]^d.
\end{aligned}$$

For later analysis, we introduce a useful lemma for the projection Q_h.

Lemma 3.4 (cf. Ye. X & Zhang S. [18]). *For any $w \in H^{r+1}$, there holds*

$$|||Q_h w - w||| \le Ch^r \|w\|_{r+1},$$

where $C > 0$ is a constant independent of h.

4. Error analysis

In this section, we will investigate the optimal error estimate for the SFWG approximation (3.1) under the weak regularity assumption (H1)-(H3). For simplicity, we will focus on the case where the symmetric positive definite coefficient tensor A in (2.1) is a piecewise constant matrix with respect to the finite element partition $\mathcal{T}_h$. The result can be extended to variable tensors without any difficulty, provided that the tensor A is piecewise sufficiently smooth.

Throughout this paper, we always use $\tilde{u}_h$ and u_h to denote the solutions of (2.4) and (3.1), respectively. Then we have $\epsilon_h := u - u_h \in V_h^0$ that

$$\epsilon_h = (u - \tilde{u}_h) + (\tilde{u}_h - Q_h\tilde{u}_h) + (Q_h\tilde{u}_h - u_h). \tag{4.1}$$

The discrete H^1 estimates of the first and second terms in (4.1) follow from Lemma 2.1 and 3.4, respectively. The L^2 estimates of the first and second terms in (4.1) follow from Lemma 2.1 and 3.2, respectively. We will focus on the error analysis of the last terms in (4.1). Define an error that

$$e_h = \{e_0, e_b\} := Q_h\tilde{u}_h - u_h, \tag{4.2}$$

where $e_0 = Q_0\tilde{u}_h - u_0$ and $e_b = Q_b\tilde{u}_h - u_b$.

Lemma 4.1. *Assume (H1) and (H2) hold. Let $\tilde{u}_h \in S_h^0 \subset H^2 \cap H_0^1$ be the standard finite element solution of* (2.4) *and $u_h \in V_h$ be the solution of SFWG-FEM in* (3.1). *Then, the error $e_h = Q_h\tilde{u}_h - u_h$ in* (4.2) *satisfies an error equation*

$$a(e_h, v_h) = a(Q_h\tilde{u}_h - \tilde{u}_h, v_h) + l_1(\tilde{u}_h, v_h) + l_2(\tilde{u}_h, v_h),\ \forall\ v_h = \{v_0, v_b\} \in V_h^0, \tag{4.3}$$

where $l_1(\tilde{u}_h, v_h)$ and $l_2(\tilde{u}_h, v_h)$ are defined as follows: for any given $w \in H^{r+1}$, define two functionals on V_h^0

$$\begin{aligned} l_1(w, v) &:= \sum_{T\in\mathcal{T}_h} \Big\langle A\big(\nabla w - \mathbb{Q}_h(\nabla w)\big) \cdot \boldsymbol{n}, Q_b v_0 - v_b \Big\rangle_{\partial T}, \\ l_2(w, v) &:= \sum_{T\in\mathcal{T}_h} \langle A\nabla w \cdot \boldsymbol{n}, v_0 - Q_b v_0 \rangle_{\partial T}, \end{aligned} \tag{4.4}$$

for any $v = \{v_0, v_b\} \in V_h^0$.

Proof. To derive an error equation, we decompose e_h into two parts since the operator $\nabla_w Q_h$ is not identical with $\mathbb{Q}_h(\nabla\phi)$. Apparently, we have a decomposition

$$e_h = (Q_h\tilde{u}_h - \tilde{u}_h) + \tilde{u}_h - u_h. \tag{4.5}$$

Then we focus on the term $a(\tilde{u}_h - u_h, v_h)$.

For any $v_h = \{v_0, v_b\} \in V_h^0$, from Lemma 3.3 and Definition 3.2, it follows that

$$\begin{aligned}
a(\tilde{u}_h, v_h) &= \sum_{T\in\mathcal{T}_h} (A\nabla_w\tilde{u}_h, \nabla_w v_h)_T = \sum_{T\in\mathcal{T}_h} \big(A\mathbb{Q}_h(\nabla\tilde{u}_h), \nabla_w v_h\big)_T \\
&= \sum_{T\in\mathcal{T}_h} \big(A\mathbb{Q}_h(\nabla\tilde{u}_h), \nabla v_0\big)_T + \sum_{T\in\mathcal{T}_h} \big\langle A\mathbb{Q}_h(\nabla\tilde{u}_h)\cdot\boldsymbol{n}, Q_b(v_b - v_0)\big\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} (A\nabla\tilde{u}_h, \nabla v_0)_T - \sum_{T\in\mathcal{T}_h} \big\langle A\mathbb{Q}_h(\nabla\tilde{u}_h)\cdot\boldsymbol{n}, Q_b v_0 - v_b\big\rangle_{\partial T}.
\end{aligned} \tag{4.6}$$

For each $v_h = \{v_0, v_b\} \in V_h^0$, $a(u_h, v_h) = (f, v_0)$ holds. Notice that we have $v_0|_T = P_r(T) \subset S_h^0|_T$ for each element $T \in \mathcal{T}_h$ and $1 \le r \le 5$. Testing Eq.(2.5) with v_0 and using Green's formula, we have

$$\begin{aligned}
(f, v_0) &= \sum_{T\in\mathcal{T}_h} \big(P_h f, v_0\big)_T \\
&= \sum_{T\in\mathcal{T}_h} \Big(-P_h\big(\nabla\cdot(A\nabla\tilde{u}_h)\big), v_0\Big)_T \\
&= \sum_{T\in\mathcal{T}_h} \big(A\nabla\tilde{u}_h, \nabla v_0\big)_T - \sum_{T\in\mathcal{T}_h} \big\langle A\nabla\tilde{u}_h\cdot\boldsymbol{n}, v_0 - v_b\big\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} \big(A\nabla\tilde{u}_h, \nabla v_0\big)_T - \sum_{T\in\mathcal{T}_h} \big\langle A\nabla\tilde{u}_h\cdot\boldsymbol{n}, Q_b v_0 - v_b\big\rangle_{\partial T} \\
&\quad - \sum_{T\in\mathcal{T}_h} \big\langle A\nabla\tilde{u}_h\cdot\boldsymbol{n}, v_0 - Q_b v_0\big\rangle_{\partial T},
\end{aligned} \tag{4.7}$$

where the second last equality follows from the fact that

$$\sum_{T\in\mathcal{T}_h} \big\langle A\nabla\tilde{u}_h\cdot\boldsymbol{n}, v_b\big\rangle_{\partial T} = 0.$$

Combining (3.1) and (4.4)-(4.7), we obtain the error equation (4.3).

Lemma 4.2. *(Trace Inequality) Let T be an element with its edge $e \in \mathcal{E}_h$. For any function $\psi \in H^1(T)$, then we have*

$$\|\psi\|_e^2 \le C(h_T^{-1}\|\psi\|_T^2 + h_T\|\nabla\psi\|_T^2). \tag{4.8}$$

Lemma 4.3. *(Inverse Inequality) There exists a constant $C = C(r)$ such that*

$$\|\nabla\psi\|_T \le C(r)h_T^{-1}\|\psi\|_T, \qquad \forall T \in \mathcal{T}_h, \tag{4.9}$$

for any polynomial ψ of degree r on T.

Lemma 4.4 (cf. Ye. X & Zhang S. [18]). *For any $w \in H^{r+1}$ and $v = \{v_0, v_b\} \in V_h^0$, there hold*

$$|l_1(w,v)| \le Ch^r \|w\|_{r+1} |||v|||,$$
$$|l_2(w,v)| \le Ch^r \|w\|_{r+1} |||v|||,$$

where $C > 0$ is a constant independent of h.

Applying the error equation (4.3), we derive the optimal order discrete H^1-norm error estimate of e_h in (4.2).

Theorem 4.1. *Assume that (H1) and (H2) hold. Let $\tilde{u}_h \in S_h^0 \subset H^2 \cap H_0^1$ be the standard finite element solution of* (2.4) *and $u_h \in V_h$ be the stabilizer free weak Galerkin finite element solution of* (3.1). *Then, for $e_h = Q_h\tilde{u}_h - u_h$, there exists a constant $C > 0$ independent of h such that*

$$|||e_h||| \le Ch\|\tilde{u}_h\|_2.$$

Proof. Taking $v_h = e_h$ in error equation (4.3) yields

$$a(e_h, e_h) = a(Q_h\tilde{u}_h - \tilde{u}_h, e_h) + l_1(\tilde{u}_h, e_h) + l_2(\tilde{u}_h, e_h).$$

According to Lemma 3.1, 3.4 and 4.4, we get

$$\begin{aligned}
|||e_h|||^2 &\le |a(Q_h\tilde{u}_h - \tilde{u}_h, e_h)| + |l_1(\tilde{u}_h, e_h)| + |l_2(\tilde{u}_h, e_h)| \\
&\le \beta |||Q_h\tilde{u}_h - \tilde{u}_h||| \, |||e_h||| + Ch\|\tilde{u}_h\|_2 |||e_h||| + Ch\|\tilde{u}_h\|_2 |||e_h||| \\
&\le Ch\|\tilde{u}_h\|_2 |||e_h|||,
\end{aligned}$$

which leads to the desired estimate.

Let us derive the optimal order L^2 error estimate of e_0 by the standard duality argument. The dual equation of (2.5) is seeking $\Phi \in H_0^1 \cap H^2$ such that

$$-P_h\big(\nabla \cdot (A\nabla\Phi)\big) = P_h e_0, \tag{4.10}$$

where $e_0 = Q_0\tilde{u}_h - u_0$ as we defined in (4.2), and it follows from the elliptic regularity that [6],

$$\|\Phi\|_2 \le C\|e_0\|. \tag{4.11}$$

Theorem 4.2. *Assume that (H1) and (H2) hold. Let $\tilde{u}_h \in S_h^0 \subset H^2 \cap H_0^1$ be the solution of standard FEM in* (2.4) *and $u_h \in V_h$ be the solution of SFWG-FEM in* (3.1). *Then for $e_0 = Q_0\tilde{u}_h - u_0$, there exists a constant $C > 0$ independent of h such that*

$$\|e_0\| \le Ch^2\|\tilde{u}_h\|_2.$$

Proof. Testing Eq. (4.10) by e_0 yields

$$\begin{aligned}\|e_0\|^2 &= \sum_{T\in\mathcal{T}_h} \big(e_0, e_0\big)_T \\ &= \sum_{T\in\mathcal{T}_h} \big(-\nabla\cdot(A\nabla\Phi), e_0\big)_T \\ &= \sum_{T\in\mathcal{T}_h} \big(A\nabla\Phi, \nabla e_0\big)_T - \sum_{T\in\mathcal{T}_h} \big\langle A\nabla\Phi\cdot\boldsymbol{n}, e_0 - e_b\big\rangle_{\partial T}, \\ &= \sum_{T\in\mathcal{T}_h} \big(A\nabla\Phi, \nabla e_0\big)_T - \sum_{T\in\mathcal{T}_h} \big\langle A\nabla\Phi\cdot\boldsymbol{n}, Q_b e_0 - e_b\big\rangle_{\partial T} \\ &\quad - \sum_{T\in\mathcal{T}_h} \big\langle A\nabla\Phi\cdot\boldsymbol{n}, e_0 - Q_b e_0\big\rangle_{\partial T},\end{aligned}$$

where we utilize the information for the last second inequality that

$$\sum_{T\in\mathcal{T}_h} \big\langle A\nabla\Phi\cdot\boldsymbol{n}, e_b\big\rangle_{\partial T} = 0.$$

Combining (3.1) with (4.5)-(4.7), we reach

$$\begin{aligned}\|e_0\|^2 &= \sum_{T\in\mathcal{T}_h} \big(A\nabla_w\Phi, \nabla_w e_h\big)_T - \sum_{T\in\mathcal{T}_h} \Big\langle A\big(\nabla\Phi - \mathbb{Q}_h(\nabla\Phi)\big)\cdot\boldsymbol{n}, Q_b e_0 - e_b\Big\rangle_{\partial T} \\ &\quad - \sum_{T\in\mathcal{T}_h} \big\langle A\nabla\Phi\cdot\boldsymbol{n}, e_0 - Q_b e_0\big\rangle_{\partial T} \\ &= a(e_h,\Phi) - l_1(\Phi, e_h) - l_2(\Phi, e_h)\end{aligned} \tag{4.12}$$

Let us bound the term $|a(e_h,\Phi)|$ first,

$$\begin{aligned} a(e_h,\Phi) &= a(Q_h\tilde{u}_h - \tilde{u}_h, \Phi) + a(\tilde{u}_h - u_h, Q_h\Phi) + a(\tilde{u}_h - u_h, \Phi - Q_h\Phi) \\ &= a(Q_h\tilde{u}_h - \tilde{u}_h, \Phi) + l_1(\tilde{u}_h, Q_h\Phi) + l_2(\tilde{u}_h, Q_h\Phi) + a(\tilde{u}_h - u_h.\Phi - Q_h\Phi).\end{aligned} \tag{4.13}$$

In the following, we analyze these four terms in (4.13) successively. For the first term, we find the intermediate term $\big(\nabla_w(Q_h\tilde{u}_h - \tilde{u}_h), A\mathbb{Q}_0(\nabla\Phi)\big)_T = 0$, which comes from the Definition 3.2, the projection $\mathbb{Q}_0$ and Green's formula

$$\begin{aligned}&\big(\nabla_w(Q_h\tilde{u}_h - \tilde{u}_h), A\mathbb{Q}_0(\nabla\Phi)\big)_T \\ &= \big(\nabla(Q_0\tilde{u}_h - \tilde{u}_h), A\mathbb{Q}_0(\nabla\Phi)\big)_T + \Big\langle Q_b\big((Q_b\tilde{u}_h - \tilde{u}_h) - (Q_0\tilde{u}_h - \tilde{u}_h)\big), A\mathbb{Q}_0(\nabla\Phi)\cdot\boldsymbol{n}\Big\rangle_{\partial T} \\ &= \big(\nabla(Q_0\tilde{u}_h - \tilde{u}_h), A\mathbb{Q}_0(\nabla\Phi)\big)_T + \big\langle Q_b\tilde{u}_h - Q_0\tilde{u}_h, A\mathbb{Q}_0(\nabla\Phi)\cdot\boldsymbol{n}\big\rangle_{\partial T} \\ &= -\Big((Q_0\tilde{u}_h - \tilde{u}_h), \nabla\cdot\big(A\mathbb{Q}_0(\nabla\Phi)\big)\Big)_T + \Big\langle (Q_0\tilde{u}_h - \tilde{u}_h) + (Q_b\tilde{u}_h - Q_0\tilde{u}_h), A\mathbb{Q}_0(\nabla\Phi)\cdot\boldsymbol{n}\Big\rangle_{\partial T} \\ &= \big\langle Q_b\tilde{u}_h - \tilde{u}_h, A\mathbb{Q}_0(\nabla\Phi)\cdot\boldsymbol{n}\big\rangle_{\partial T} = 0.\end{aligned}$$

Applying the above equation, Lemma 3.2, 3.3, 3.4 and Cauchy-Schwarz inequality yields

$$
\begin{aligned}
&|a(Q_h\tilde{u}_h-\tilde{u}_h,\Phi)|\\
=&\left|\sum_{T\in\mathcal{T}_h}\big(A\nabla_w(Q_h\tilde{u}_h-\tilde{u}_h),\nabla_w\Phi\big)_T\right|\\
\le&\sum_{T\in\mathcal{T}_h}\left|\big(A\nabla_w(Q_h\tilde{u}_h-\tilde{u}_h),\mathbb{Q}_h(\nabla\Phi)\big)_T\right|\\
=&\sum_{T\in\mathcal{T}_h}\left|\big(\nabla_w(Q_h\tilde{u}_h-\tilde{u}_h),A\nabla\Phi\big)_T\right|\\
=&\sum_{T\in\mathcal{T}_h}\left|\Big(\nabla_w(Q_h\tilde{u}_h-\tilde{u}_h),A\big(\nabla\Phi-\mathbb{Q}_0(\nabla\Phi)\big)\Big)_T\right|\\
\le&C\left(\sum_{T\in\mathcal{T}_h}\|\nabla_w(Q_h\tilde{u}_h-\tilde{u}_h)\|_T^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|\nabla\Phi-\mathbb{Q}_0(\nabla\Phi)\|_T^2\right)^{1/2}\\
\le&Ch{|\kern-0.25ex|\kern-0.25ex|}Q_h\tilde{u}_h-\tilde{u}_h{|\kern-0.25ex|\kern-0.25ex|}\|\Phi\|_2\\
\le&Ch^2\|\tilde{u}_h\|_2\|\Phi\|_2.
\end{aligned}
\tag{4.14}
$$

For the second term, by means of Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&|l_1(\tilde{u}_h,Q_h\Phi)|\\
=&\left|\sum_{T\in\mathcal{T}_h}\Big\langle A\big(\nabla\tilde{u}_h-\mathbb{Q}_h(\nabla\tilde{u}_h)\big)\cdot\boldsymbol{n},Q_b(Q_0\Phi)-Q_b\Phi\Big\rangle_{\partial T}\right|\\
\le&\sum_{T\in\mathcal{T}_h}\left|\Big\langle A\big(\nabla\tilde{u}_h-\mathbb{Q}_h(\nabla\tilde{u}_h)\big)\cdot\boldsymbol{n},Q_b(Q_0\Phi-\Phi)\Big\rangle_{\partial T}\right|\\
\le&C\sum_{T\in\mathcal{T}_h}\|\nabla\tilde{u}_h-\mathbb{Q}_h(\nabla\tilde{u}_h)\|_{\partial T}\|Q_0\Phi-\Phi\|_{\partial T}\\
\le&C\left(\sum_{T\in\mathcal{T}_h}\|\nabla\tilde{u}_h-\mathbb{Q}_h(\nabla\tilde{u}_h)\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi-\Phi\|_{\partial T}^2\right)^{1/2}.
\end{aligned}
$$

Based on Lemma 3.2 combined with the trace inequality (4.8) and the inverse inequality (4.9), it is evident that

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h}\|\nabla\tilde{u}_h-\mathbb{Q}_h(\nabla\tilde{u}_h)\|_{\partial T}^2&\le Ch\|\tilde{u}_h\|_2^2,\\
\sum_{T\in\mathcal{T}_h}\|Q_0\Phi-\Phi\|_{\partial T}^2&\le Ch^3\|\Phi\|_2^2.
\end{aligned}
$$

These two estimates give

$$
|l_1(\tilde{u}_h,Q_h\Phi)|\le Ch^2\|\tilde{u}_h\|_2\|\Phi\|_2. \tag{4.15}
$$

For the third term, utilizing the projection operator $\mathbb{Q}_{r-1}$ and Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&|l_2(\tilde{u}_h, Q_h\Phi)|\\
=&\left|\sum_{T\in\mathcal{T}_h}\langle A\nabla\tilde{u}_h\cdot\boldsymbol{n}, Q_0\Phi - Q_b(Q_0\Phi)\rangle_{\partial T}\right|\\
\leq&\sum_{T\in\mathcal{T}_h}\left|\Big\langle A\big(\nabla\tilde{u}_h-\mathbb{Q}_{r-1}(\nabla\tilde{u}_h)\big)\cdot\boldsymbol{n}, Q_0\Phi - Q_b(Q_0\Phi)\Big\rangle_{\partial T}\right|\\
\leq& C\sum_{T\in\mathcal{T}_h}\|\nabla\tilde{u}_h-\mathbb{Q}_{r-1}(\nabla\tilde{u}_h)\|_{\partial T}\|Q_0\Phi - Q_b(Q_0\Phi)\|_{\partial T}\\
\leq& C\left(\sum_{T\in\mathcal{T}_h}\|\nabla\tilde{u}_h-\mathbb{Q}_{r-1}(\nabla\tilde{u}_h)\|^2_{\partial T}\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi - Q_b(Q_0\Phi)\|^2_{\partial T}\right)^{1/2}.
\end{aligned}
$$

From the trace inequality (4.8) and inverse inequality (4.9) , it follows that

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h}\|\nabla\tilde{u}_h-\mathbb{Q}_{r-1}(\nabla\tilde{u}_h)\|^2_{\partial T}&\leq Ch\|\tilde{u}_h\|_2^2,\\
\sum_{T\in\mathcal{T}_h}\|Q_0\Phi - Q_b(Q_0\Phi)\|^2_{\partial T}&\leq Ch^3\|\Phi\|_2^2.
\end{aligned}
$$

These two estimates lead to

$$
|l_2(\tilde{u}_h, Q_h\Phi)|\leq Ch^2\|\tilde{u}_h\|_2\|\Phi\|_2. \tag{4.16}
$$

For the fourth term, according to Lemma 3.1, 3.4 and Theorem 4.1, we have

$$
\begin{aligned}
&|a(\tilde{u}_h-u_h, \Phi - Q_h\Phi)|\\
\leq&\beta{|\!|\!|}\tilde{u}_h-u_h{|\!|\!|}\,{|\!|\!|}\Phi-Q_h\Phi{|\!|\!|}\\
\leq& C({|\!|\!|}\tilde{u}_h-Q_h\tilde{u}_h{|\!|\!|}+{|\!|\!|}e_h{|\!|\!|})\cdot{|\!|\!|}\Phi-Q_h\Phi{|\!|\!|}\\
\leq& Ch^2\|\tilde{u}_h\|_2\|\Phi\|_2.
\end{aligned} \tag{4.17}
$$

Combined the estimates (4.14)-(4.17) together, we arrive at

$$
|a(e_h,\Phi)|\leq Ch^2\|\tilde{u}_h\|_2\|\Phi\|_2. \tag{4.18}
$$

Now, applying Lemma 4.4 and Theorem 4.1, we have

$$
|l_1(\Phi, e_h)|\leq Ch\|\Phi\|_2{|\!|\!|}e_h{|\!|\!|}\leq Ch^2\|\tilde{u}_h\|_2\|\Phi\|_2, \tag{4.19}
$$

$$
|l_2(\Phi, e_h)|\leq Ch\|\Phi\|_2{|\!|\!|}e_h{|\!|\!|}\leq Ch^2\|\tilde{u}_h\|_2\|\Phi\|_2. \tag{4.20}
$$

Combing (4.12), (4.11) with (4.18)-(4.20), we obtain

$$
\|e_0\|^2\leq Ch^2\|\tilde{u}_h\|_2\|\Phi\|_2\leq Ch^2\|\tilde{u}_h\|_2\|e_0\|,
$$

which leads to the desired estimate.

In the end of this section, we obtain the optimal order discrete H^1 and L^2 error estimate for the SFWG approximation (3.1).

Theorem 4.3. *Assume that (H1)-(H3) hold true and the exact solution $u \in H^{1+s}(\Omega)$ $(0 < s \leq 1)$. Let $u_h \in V_h$ be the solution of SFWG-FEM in* (3.1). *Then there exists a constant $C > 0$ independent of h such that*

$$|||u - u_h||| \leq Ch^s \|f\|.$$

Proof. According to (4.1), Lemma 2.1, 3.4 and Theorem 4.1, we obtain

$$\begin{aligned} |||u - u_h||| &\leq |||u - \tilde{u}_h||| + |||\tilde{u}_h - Q_h\tilde{u}_h||| + |||Q_h\tilde{u}_h - u_h||| \\ &\leq C\|\nabla u - \nabla \tilde{u}_h\| + Ch\|\tilde{u}_h\|_2 + Ch\|\tilde{u}_h\|_2 \\ &\leq Ch^s\|f\|. \end{aligned}$$

The proof is now completed.

Theorem 4.4. *Assume that (H1)-(H3) hold true and the exact solution $u \in H^{1+s}(\Omega)$ $(0 < s \leq 1)$. Let $u_h = \{u_0, u_b\} \in V_h$ be the stabilizer free weak Galerkin finite element solution of* (3.1). *Then there exists a constant $C > 0$ independent of h such that*

$$\|u - u_0\| \leq Ch^{1+s}\|f\|.$$

Proof. According to (4.1), Lemma 2.1, 3.2 and Theorem 4.2, we obtain

$$\begin{aligned} \|u - u_0\| &\leq \|u - \tilde{u}_h\| + \|\tilde{u}_h - Q_0\tilde{u}_h\| + \|Q_0\tilde{u}_h - u_0\| \\ &\leq Ch^{1+s}\|f\| + Ch^2\|\tilde{u}_h\|_2 + Ch^2\|\tilde{u}_h\|_2 \\ &\leq Ch^{1+s}\|f\|. \end{aligned}$$

This completes the proof.

5. Numerical Experiments

The goal of this section is to numerically verify the H^{1+s} convergence theory $(0 < s \leq 1)$ for the stabilizer free weak Galerkin finite element method through three numerical examples. In particular, we consider elliptic equations with various lower regularity and illustrate the corresponding convergence rate. Let $\Omega = [0,1] \times [0,1]$, and we use uniform triangular meshes $\mathcal{T}_h$ where h denotes the spatial mesh size. Then the $(P_1(T), P_0(e), [P_2(T)]^2)$ finite element space is

$$V_h := \{v = \{v_0, v_b\} : v_0 \in P_1(T), v_b \in P_0(e), \nabla_w v \in [P_2(T)]^2, \forall T \in \mathcal{T}_h, \forall e \subset \partial T\}.$$

The first example is presented to show the standard convergence rate for H^2-regularity problem. The second and the third are supplied for studying the reliability of the SFWG method for H^{1+s} $(0 < s \leq 1)$ problems with various regularity. Gaussian quadratures are used in the numerical integration.

Example 5.1. Consider (2.1) with the diffusion coefficient matrix

$$A = \begin{pmatrix} x^2+y^2+1 & 0 \\ 0 & x^2+y^2+1 \end{pmatrix}.$$

We choose the source term f and the boundary value g such that the analytic solution is

$$u = \sin(\pi x)\sin(\pi y).$$

Fig.1 shows the profile of the exact solution on the left and the stabilizer free weak Galerkin approximation on the right. We can find that the scheme is accurate. Table 1 shows that the SFWG-FEM errors and the convergence rates of Example 5.1 in L^2-norm and $|||\cdot|||$-norm for u. The convergence rates are of order $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$ respectively because the elliptic regularity is H^2, which is consistent with the standard SFWG-FEM.

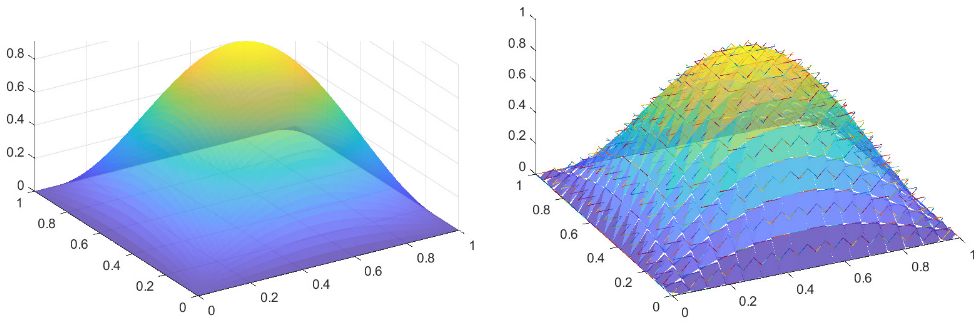

Figure 1: The profile of exact solution (left) and SFWG-FEM solution (right) of Example 5.1.

Table 1: Errors and convergence rates of Example 5.1

h	$\lvert\lvert\lvert Q_hu-u_h\rvert\rvert\rvert$	convergence rate	$\lVert Q_hu-u_h\rVert$	convergence rate
1/8	3.6124E-01	–	7.0094E-03	–
1/16	1.8130E-01	0.9946	1.7664E-03	1.9884
1/32	9.0695E-02	0.9993	4.4252E-04	1.9970
1/64	4.5344E-02	1.0001	1.1069E-04	1.9992
1/128	2.2669E-02	1.0002	2.7676E-05	1.9998

Example 5.2. Consider the problem (2.1) where the solution lies in the space H^{1+s}. Let the diffusion coefficient matrix $A = I$ where I is a 2×2 identity matrix. We select the boundary value g and the source term f so that the analytical solution is

$$u = x(1-x)y(1-y)(\sqrt{x^2+y^2})^{-2+\gamma},\ 0<\gamma\le 1.$$

For any $0<\epsilon<\gamma$, let $s=\gamma-\epsilon>0$. Then, $u\in H_0^1\cap H^{1+s}$, $0<s<1$.

We use the uniform triangular grids as before and carry out numerical computation with different step-sizes. For this example, a 32×32 mesh is used for plotting Fig.2. The profile of exact solution is presented in Fig.2 (left) and the solution calculated by the SFWG-FEM is shown in Fig.2 (right). We can find that the singular point is $(0,0)$.

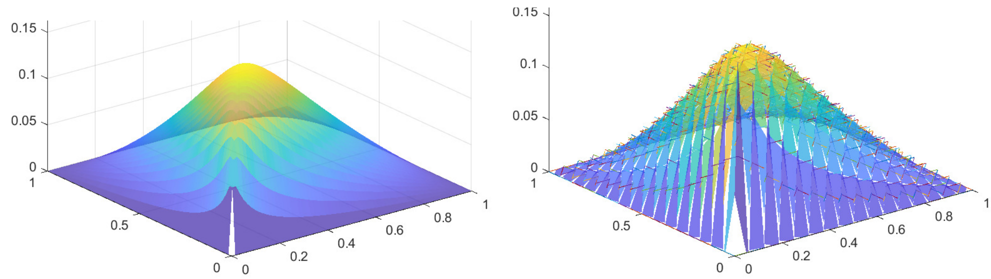

Figure 2: The profile of exact solution (left) and SFWG-FEM solution (right) of Example 5.2 as $\gamma = 0.5$.

Since the solution $u \in H^1 \cap H^{1+(\gamma-\epsilon)}$, as proved in the two theorems, we have the $\mathcal{O}(h^s)$ convergence in $|||\cdot|||$-norm and $\mathcal{O}(h^{1+s})$ convergence in L^2-norm, which are verified by our computational results. Table 2-4 show that the errors and convergence rates of Example 5.2 as $\gamma = 0.25, \gamma = 0.5$ and $\gamma = 0.75$. We can find that the H^1-convergence rate is about of $\mathcal{O}(h^{0.25})$, $\mathcal{O}(h^{0.5})$ and $\mathcal{O}(h^{0.7})$, with L^2-convergence rate being $\mathcal{O}(h^{1.25})$, $\mathcal{O}(h^{1.5})$ and $\mathcal{O}(h^{1.7})$, respectively.

The numerical results are consistent with the theory for these cases and demonstrate that the SFWG-FEM method is also accurate and robust when the elliptic regularity is H^{1+s} $(0 < s \leq 1)$, and a higher regularity leads to a faster convergence rate.

Table 2: Errors and convergence rates of Example 5.2 with $\gamma = 0.25$

h	$\lvert\lvert\lvert Q_h u - u_h \rvert\rvert\rvert$	convergence rate	$\lVert Q_h u - u_h \rVert$	convergence rate
1/8	3.1102E-01	–	8.5605E-03	–
1/16	2.6202E-01	0.2473	3.6025E-03	1.2487
1/32	2.2025E-01	0.2505	1.5102E-03	1.2542
1/64	1.8508E-01	0.2509	6.3335E-04	1.2537
1/128	1.5556E-01	0.2507	2.6586E-04	1.2523

Example 5.3. Consider (2.1) with the diffusion coefficient matrix

$$A = \begin{pmatrix} x^2+y^2+1 & xy \\ xy & x^2+y^2+1 \end{pmatrix}.$$

We select the boundary value g and the source term f so that the analytical solution fits in with the one of the Example 5.2, which is

$$u = x(1-x)y(1-y)(\sqrt{x^2+y^2})^{-2+\gamma}, \, 0 < \gamma \leq 1.$$

For any $0 < \epsilon < \gamma$, let $s = \gamma - \epsilon > 0$. Then, $u \in H_0^1 \cap H^{1+s}$, $0 < s < 1$.

Table 3: Errors and convergence rates of Example 5.2 with $\gamma = 0.5$

h	$\|\|\|Q_h u - u_h\|\|\|$	convergence rate	$\|\|Q_h u - u_h\|\|$	convergence rate
1/8	1.3165E-01	–	3.5534E-03	–
1/16	9.4204E-02	0.4828	1.3075E-03	1.4424
1/32	6.6960E-02	0.4925	4.7195E-04	1.4701
1/64	4.7469E-02	0.4963	1.6873E-04	1.4839
1/128	3.3612E-02	0.4979	6.0019E-05	1.4912

Table 4: Errors and convergence rates of Example 5.2 with $\gamma = 0.75$

h	$\|\|\|Q_h u - u_h\|\|\|$	convergence rate	$\|\|Q_h u - u_h\|\|$	convergence rate
1/8	6.8196E-02	–	2.0189E-03	–
1/16	4.3163E-02	0.6599	6.8783E-04	1.5535
1/32	2.6779E-02	0.6887	2.2254E-04	1.6279
1/64	1.6409E-02	0.7066	6.9931E-05	1.6701
1/128	9.9693E-03	0.7189	2.1579E-05	1.6963

We choose a symmetric positive definite diffusion coefficient matrix A and the solution lies in the space $H^{1+(\gamma-\epsilon)}$. Fig.3 (left) and Fig.3 (right) show the convergence order of SFWG-FEM as $u \in H^{1.5-\epsilon}$ with $0 < \epsilon < 0.5$ and $u \in H^{1.75-\epsilon}$ with $0 < \epsilon < 0.75$, respectively. The blue dash line represents the convergence of $|||\cdot|||$-norm, the red dash line is of L^2-norm. The red line in Fig.3 (left) is of slope 1.5 to be the reference of $\mathscr{O}(h^{1.5-\epsilon})$ and the blue line is of slope 0.5 to be the reference of $\mathscr{O}(h^{0.5-\epsilon})$. The lines in Fig.3 (right) are similar.

We get the error and the convergence rate with respect to the mesh-size in Table 5-6, which also conforms well the theoretical analysis. This numerical experiment shows that the SFWG-FEM is still flexible and effective though the solution has low regularity.

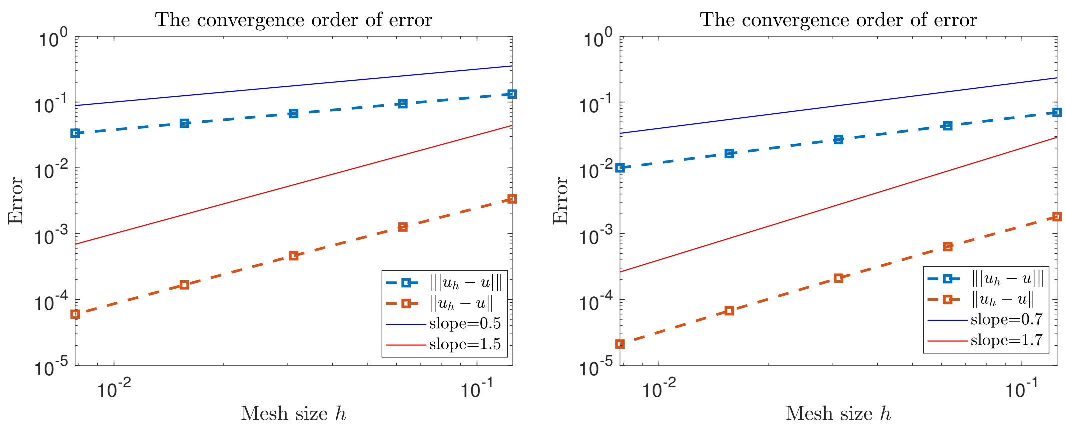

Figure 3: The convergence order of SFWG-FEM as $\gamma = 0.5$ (left) and $\gamma = 0.75$ (right).

Table 5: Errors and convergence rates of Example 5.3 with $\gamma = 0.5$

h	$\vert\vert\vert Q_h u - u_h \vert\vert\vert$	convergence rate	$\Vert Q_h u - u_h \Vert$	convergence rate
1/8	13210E-01	–	3.3673E-03	–
1/16	9.4264E-02	0.4869	1.2620E-03	1.4159
1/32	6.6955E-02	0.4935	4.6173E-04	1.4506
1/64	4.7462E-02	0.4964	1.6660E-04	1.4706
1/128	3.3608E-02	0.4979	5.9594E-05	1.4831

Table 6: Errors and convergence rates of Example 5.3 with $\gamma = 0.75$

h	$\vert\vert\vert Q_h u - u_h \vert\vert\vert$	convergence rate	$\Vert Q_h u - u_h \Vert$	convergence rate
1/8	6.9643E-02	–	1.8109E-03	–
1/16	4.3626E-02	0.6748	6.3529E-04	1.5112
1/32	2.6943E-02	0.6952	2.1055E-04	1.5932
1/64	1.6473E-02	0.7099	6.7315E-05	1.6452
1/128	9.9948E-03	0.7208	2.1021E-05	1.6791

6. Conclusion

In this paper, we have proposed and proved the error estimates in both discrete H^1-norm and standard L^2-norm for the stabilizer free weak Galerkin (SFWG) finite element methods for the second order elliptic problems with low regularity solutions. This analysis helps to remove the compulsory H^2-smoothness assumption for the real solution in all the related literature. We assume that the solution of elliptic problems has H^{1+s}-regularity, and we have proved that the H^1-convergence rate and the L^2-convergence rate are of s-order and $(1+s)$-order, respectively. The $(P_k(T), P_{k-1}(e), [P_{k+1}(T)]^d)$ elements are employed for SFWG-FEM. Our strategy is taking H^2 finite elements as the intermediate approximation. We approximate the elliptic problem using the traditional FEM with at least H^2 smooth bases, then approximate this standard finite element solution via SFWG-FEM. Therefore, the $(P_1(T), P_0(e), [P_2(T)]^d)$ elements are reliable to construct weak Galerkin finite element bases for the low-regularity problems. We also give three numerical examples for this.

Our result is an important supplementary for the new stabilizer free weak Galerkin finite element method theory. We believe this result can be extended to more types of PDEs that the finite element method can be applied to and more general meshes.

Acknowledgments

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References

[1] A. Al-Taweel, S. Hussain, R. Lin, and P. Zhu. A stabilizer free weak Galerkin finite element method for general second-order elliptic problem. *Int. J. Numer. Anal. Model.*, 18(3):311–323, 2021.

[2] A. Al-Taweel, S. Hussain, and X. Wang. A stabilizer free weak Galerkin finite element method for parabolic equation. *J. Comput. Appl. Math.*, 392:Paper No. 113373, 12, 2021.

[3] A. Al-Taweel and X. Wang. A note on the optimal degree of the weak gradient of the stabilizer free weak Galerkin finite element method. *Appl. Numer. Math.*, 150:444–451, 2020.

[4] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[5] S. Chai, Y. Wang, W. Zhao, and Y. Zou. A C^0 weak Galerkin method for linear Cahn-Hilliard-Cook equation with random initial condition. *Appl. Math. Comput.*, 414:Paper No. 126659, 11, 2022.

[6] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[7] L. Mu, J. Wang, and X. Ye. A weak Galerkin finite element method with polynomial reduction. *J. Comput. Appl. Math.*, 285:45–58, 2015.

[8] L. Mu, J. Wang, and X. Ye. Weak Galerkin finite element methods on polytopal meshes. *Int. J. Numer. Anal. Model.*, 12(1):31–53, 2015.

[9] L. Mu, J. Wang, X. Ye, and S. Zhang. A weak Galerkin finite element method for the Maxwell equations. *J. Sci. Comput.*, 65(1):363–386, 2015.

[10] H. Peng and Q. Zhai. Weak Galerkin method for the Stokes equations with damping. *Discrete Contin. Dyn. Syst. Ser. B*, 27(4):1853–1875, 2022.

[11] J. Shields, S.and Li and E. A. Machorro. Weak Galerkin methods for time-dependent Maxwell's equations. *Comput. Math. Appl.*, 74(9):2106–2124, 2017.

[12] T. Tian, Q. Zhai, and R. Zhang. A new modified weak Galerkin finite element scheme for solving the stationary Stokes equations. *J. Comput. Appl. Math.*, 329:268–279, 2018.

[13] J. Wang and X. Ye. A weak Galerkin finite element method for second-order elliptic problems. *J. Comput. Appl. Math.*, 241:103–115, 2013.

[14] J. Wang and X. Ye. A weak Galerkin mixed finite element method for second order elliptic problems. *Math. Comp.*, 83(289):2101–2126, 2014.

[15] Y. Wang, Y. Zou, and S. Chai. $(1+s)$-order convergence analysis of weak Galerkin finite element methods for second order elliptic equations. *Adv. Appl. Math. Mech.*, 13(3):554–568, 2021.

[16] X. Ye and S. Zhang. A stabilizer-free weak Galerkin finite element method on polytopal meshes. *J. Comput. Appl. Math.*, 371:112699, 9, 2020.

[17] X. Ye and S. Zhang. A stabilizer free weak Galerkin method for the biharmonic equation on polytopal meshes. *SIAM J. Numer. Anal.*, 58(5):2572–2588, 2020.

[18] X. Ye and S. Zhang. A new weak gradient for the stabilizer free weak Galerkin method with polynomial reduction. *Discrete Contin. Dyn. Syst. Ser. B*, 26(8):4131–4145, 2021.

[19] X. Ye and S. Zhang. A stabilizer-free pressure-robust finite element method for the Stokes equations. *Adv. Comput. Math.*, 47(2):Paper No. 28, 17, 2021.

[20] X. Ye and S. Zhang. A stabilizer free weak Galerkin finite element method on polytopal mesh: Part II. *J. Comput. Appl. Math.*, 394:Paper No. 113525, 11, 2021.

[21] X. Ye and S. Zhang. A stabilizer free weak Galerkin finite element method on polytopal mesh: Part III. *J. Comput. Appl. Math.*, 394:Paper No. 113538, 9, 2021.

[22] Q. Zhai, X. Hu, and R. Zhang. The shifted-inverse power weak Galerkin method for eigen-

value problems. *J. Comput. Math.*, 38(4):606–623, 2020.

[23] Q. Zhai, H. Xie, R. Zhang, and Z. Zhang. The weak Galerkin method for elliptic eigenvalue problems. *Commun. Comput. Phys.*, 26(1):160–191, 2019.

[24] H. Zhang, Y. Zou, S. Chai, and H. Yue. Weak Galerkin method with $(r, r-1, r-1)$-order finite elements for second order parabolic equations. *Appl. Math. Comput.*, 275:24–40, 2016.

[25] H. Zhang, Y. Zou, Y. Xu, Q. Zhai, and H. Yue. Weak Galerkin finite element method for second order parabolic equations. *Int. J. Numer. Anal. Model.*, 13(4):525–544, 2016.

[26] R. Zhang and Q. Zhai. A weak Galerkin finite element scheme for the biharmonic equations by using polynomials of reduced order. *J. Sci. Comput.*, 64(2):559–585, 2015.

[27] C. Zhou, Y. Zou, S. Chai, and F. Zhang. Mixed weak Galerkin method for heat equation with random initial condition. *Math. Probl. Eng.*, pages Art. ID 8796345, 11, 2020.

[28] C. Zhou, Y. Zou, S. Chai, Q. Zhang, and H. Zhu. Weak Galerkin mixed finite element method for heat equation. *Appl. Numer. Math.*, 123:180–199, 2018.

[29] H. Zhu, Y. Zou, S. Chai, and C. Zhou. Numerical approximation to a stochastic parabolic PDE with weak Galerkin method. *Numer. Math. Theory Methods Appl.*, 11(3):604–617, 2018.

[30] H. Zhu, Y. Zou, S. Chai, and C. Zhou. A weak Galerkin method with RT elements for a stochastic parabolic differential equation. *East Asian J. Appl. Math.*, 9(4):818–830, 2019.